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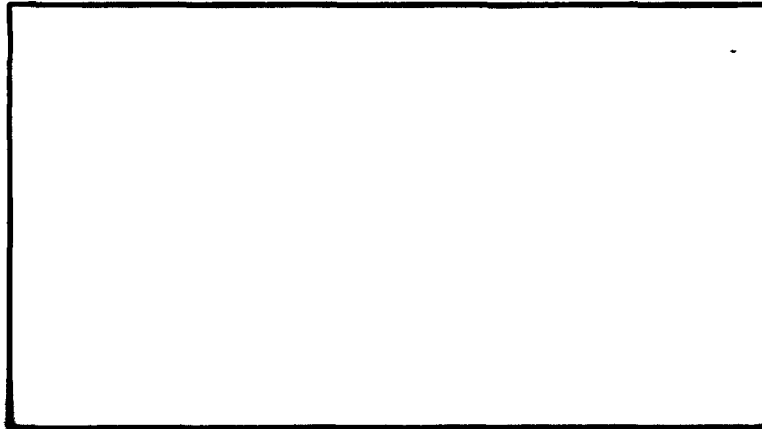
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# AIR FORCE INSTITUTE OF TECHNOLOGY



AIR UNIVERSITY  
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**INVESTIGATION  
OF RELIABILITY CHARACTERISTICS  
OF SOME COMPLEX NETWORKS**

**THESIS**

**GRE/HR/63-4    James Hurst Stephenson  
Major                      USAF**

**INVESTIGATION OF RELIABILITY CHARACTERISTICS  
OF SOME COMPLEX NETWORKS**

**THESIS**

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the Air Force Institute of Technology**

**Air University**

**in Partial Fulfillment of the  
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Preface

While the network reliability functions encountered in textbooks are justifiably kept simple, those encountered in practice are frequently quite complex. Visualizing the many hours which might be spent in duplication of effort by many engineers and students laboriously developing reliability functions for complex networks, I thought that it might be useful to do this for a few of the more common ones and make the results available to all. To this end have I written this thesis with the hope that the results contained herein might prove useful to others in the area of systems reliability.

I wish to express my gratitude for the guidance and encouragement of Prof. T. L. Regulinski, my faculty advisor for this study.

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Abstract

Reliability functions and a procedure are developed by which maximum reliable parallel-series and series-parallel networks may be designed. These networks are composed of equal elements having open and short circuit failure modes. Reliability functions are developed for bridge and lattice networks composed of unequal elements having a single failure mode. These functions are given in two forms, one suitable for substitution into more complex networks and one suitable for desk calculation and computer programming. Reliability functions are developed for two, three, four, and five station polygonal networks composed of equal paths with single failure modes. Reliability improvement is shown as stations are increased.

## I. Introduction

Network analysis is used today in many fields to study the relationship and interdependence of functionally connected elements. One of the many useful applications of network analysis is in systems reliability calculation, the use to which it is put in this thesis.

The purpose of this study was to develop expressions for the reliability of some common types of networks in terms of element reliability factors and to establish procedures for their use in calculating system, or network, reliability. A cursory review of networks in general will reveal that there are some configurations which are quite common and for which general reliability expressions might be useful. Those networks which were selected for this study were the parallel-series, the series-parallel, the bridge, the lattice, and four of the polygonal class of networks.

While it is true that system reliability is basically dependent upon the reliability of the elements of which the system is composed, it is frequently possible, by judicious arrangement of the available elements, to achieve a network reliability greater than the reliability of a single element or of any other network configuration.

Shannon and Moore have shown that it is possible to construct reliable circuits from less reliable elements and have developed reliability functions for parallel-series and series-parallel circuits assuming equal elements and single failure modes ( Ref 1:205 ). ARINC Research Corporation Publication No. 123-7-196 includes the development of expressions for determining the optimum reliability of series and parallel circuits composed of equal elements having two failure modes ( Ref 5:12-29, 12-33 ). In chapter one of this study are formulated equations with procedures for their use in designing maximum reliability parallel-series and series-parallel circuits composed of equal elements having two possible modes of failure. Thus, the work begun by Shannon and Moore and continued in the above mentioned ARINC publication has been extended one step further by this study.

The other networks in this study are assumed to be composed of unlike elements with a single failure mode and the development of their reliability expressions comprises chapters two, three, and four. Appendix A consists of a proof of the Disjunctive Expansion Theorem of Boolean Algebra.

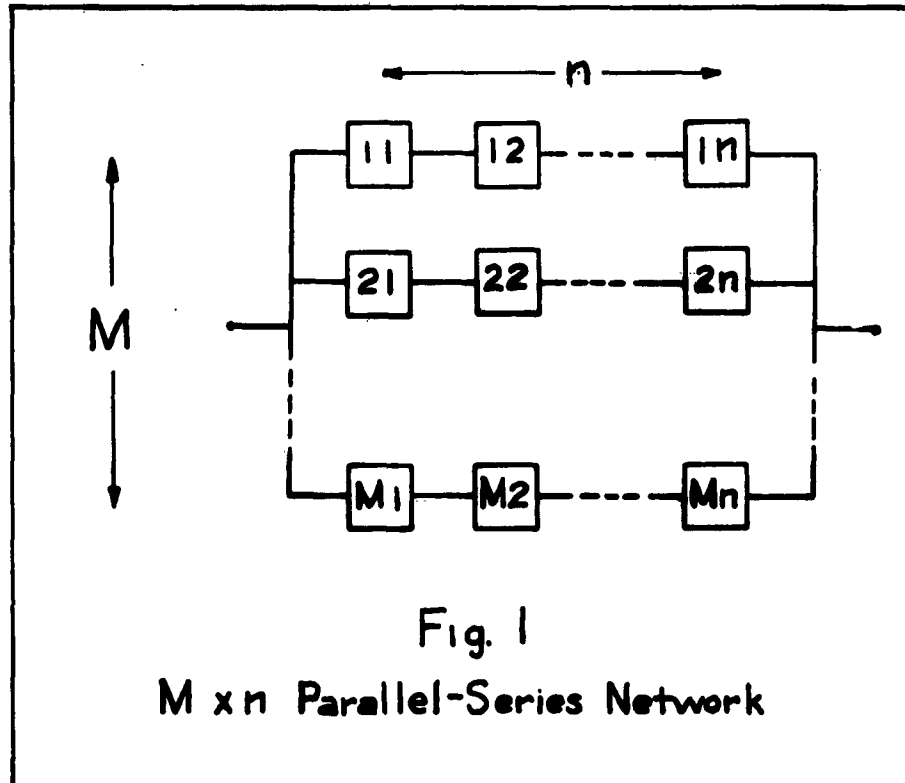
The results of the study will prove useful in both desk calculator and digital computer computation of network reliability values and in development of network reliability expressions for more complex networks.

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Because of the differences between the elements in the networks of chapter one and those in the networks of chapters two, three, and four, the work in those two areas is independent.

II. Parallel-Series and Series-Parallel Networks

The first network to be considered is the  $M \times n$  parallel-series network which is shown in Fig. 1.



Each of the elements can fail open or short and the conditions of network failure are that (a) all  $n$  elements in at least one path fail short or (b) at least one element in all  $M$  paths fails open.

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It is assumed that the probabilities of failure of the elements are independent. The event of all  $n$  elements in at least one path failing short will be called event A and the event of at least one element in all  $M$  paths failing open will be called event B. The probabilities of an element's failing short or open will be indicated by the symbols  $q_{si}$  and  $q_{oi}$  respectively.

The joint probability of all the elements in row  $i$  failing short,  $Q_{si}$ , is thus given by the product of the element probabilities of short circuit failure,

$$Q_{si} = \prod_{j=1}^n q_{sij} \quad (1)$$

Subtracting  $Q_{si}$  from 1 gives the probability that at least one element in path  $i$  does not fail short; i.e., the reliability of the path with respect to the short circuit mode of failure,

$$R_{si} = 1 - \prod_{j=1}^n q_{sij} \quad (2)$$

From this, the joint probability that no path has all elements to fail short is

$$R_s = \prod_{i=1}^M (1 - \prod_{j=1}^n q_{sij}) \quad (3)$$

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and finally, the probability of event A is seen to be

$$P(A) = 1 - \prod_{i=1}^M \left( 1 - \prod_{j=1}^n q_{sij} \right) \quad (4)$$

For equal elements this expression becomes

$$P(A) = 1 - \left[ 1 - (q_s)^n \right]^M \quad (5)$$

Considering now the open circuit mode of failure, the probability that no element in path i fails open,  $R_{oi}$ , is equal to the product of the probabilities of the element's not failing open.

$$R_{oi} = \prod_{j=1}^n (1 - q_{oij}) \quad (6)$$

The complement of this expression is the probability that at least one element in row i fails open,

$$Q_{oi} = 1 - \prod_{j=1}^n (1 - q_{oij}) \quad (7)$$

and so the probability of event B is given by

$$P(B) = \prod_{i=1}^M \left[ 1 - \prod_{j=1}^n (1 - q_{oij}) \right] \quad (8)$$

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which, for equal elements becomes

$$P(B) = [1 - (1 - q_e)^n]^M \quad (9)$$

Since the events A and B are mutually exclusive, the unreliability of the network is simply the sum of the probabilities of events A and B,

$$Q_N = P(A) + P(B) \quad (10)$$

$$Q_N = 1 - \prod_{i=1}^M (1 - \prod_{j=1}^n q_{sij}) + \prod_{i=1}^M [1 - \prod_{j=1}^n (1 - q_{eij})] \quad (11)$$

and the general expression for the parallel-series network reliability will be given by the complement of the last expression,

$$R_N = 1 - Q_N \quad (12)$$

$$R_N = \prod_{i=1}^M (1 - \prod_{j=1}^n q_{sij}) - \prod_{i=1}^M [1 - \prod_{j=1}^n (1 - q_{eij})] \quad (13)$$

It is not meaningful to speak of an optimum number of elements for a network composed of a number of different elements; however, for networks composed of equal elements it is possible to determine the number of elements which will maximize network reliability. For an equal element

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parallel-series network, Eq (13) becomes

$$R_n = [1 - (q_s)^n]^M - [1 - (1 - q_s)^n]^M \quad (14)$$

For various given values of  $q_s$  and  $q_o$ , optimum values of  $n$  (the number of elements in series) have been determined and graphed as shown in Fig. 2.

( Ref 5:12-30 ).

Since  $n$ , the optimum length of the network, has been determined and is a constant for given values of  $q_s$  and  $q_o$ , it is necessary only to find  $M$ , the optimum width of the network. This may be done by taking the derivative of  $R$  with respect to  $M$  and equating it to zero.

$$\frac{dR_n}{dM} = \frac{d}{dM} \left\{ [1 - (q_s)^n]^M - [1 - (1 - q_s)^n]^M \right\} \quad (15)$$

$$\text{Since } \frac{d}{dx} a^x = a^x \ln a,$$

$$\begin{aligned} \frac{dR_n}{dM} = & [1 - (q_s)^n]^M \ln[1 - (q_s)^n] \\ & - [1 - (1 - q_s)^n]^M \ln[1 - (1 - q_s)^n] \end{aligned} \quad (16)$$

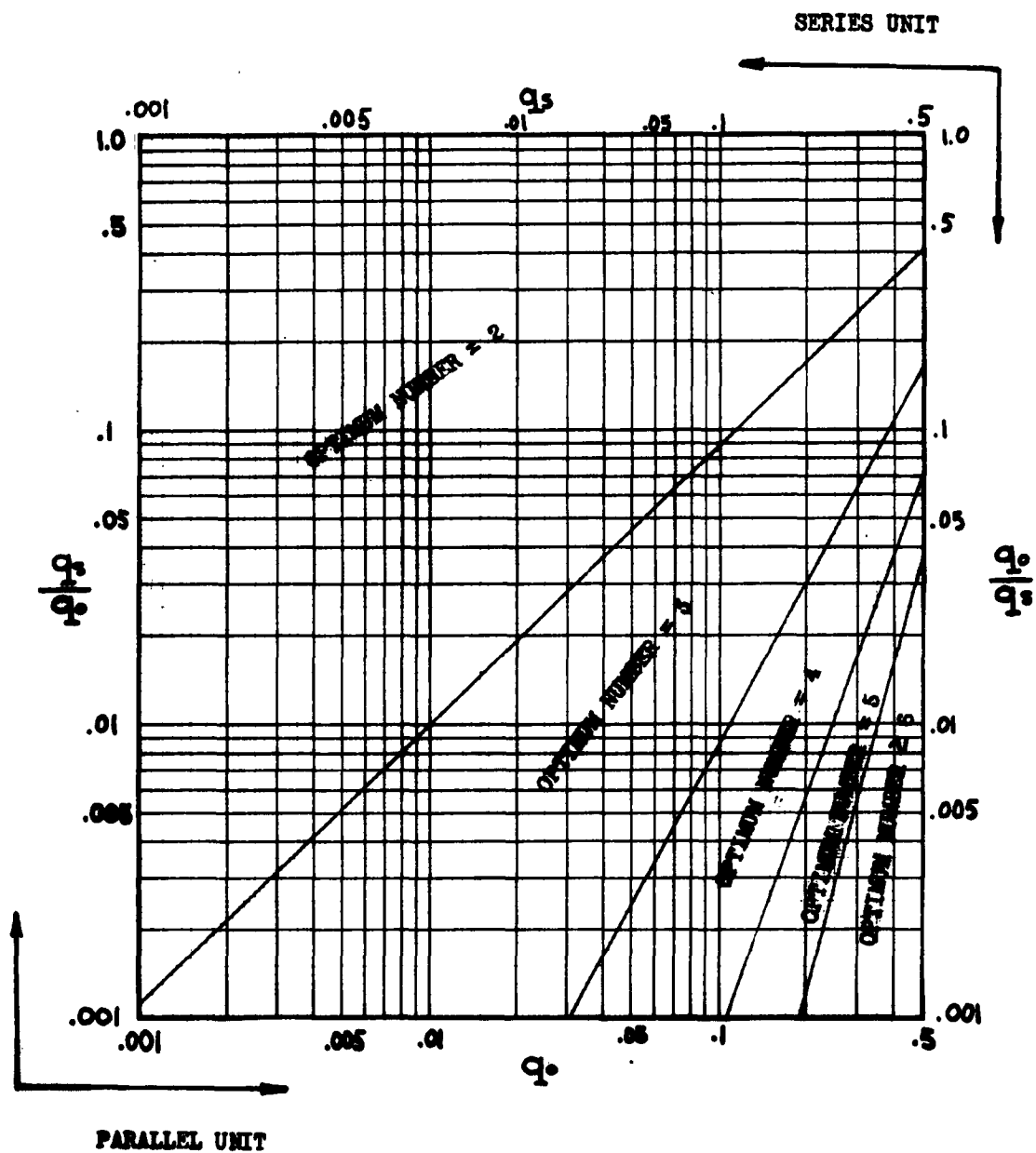


FIGURE 2

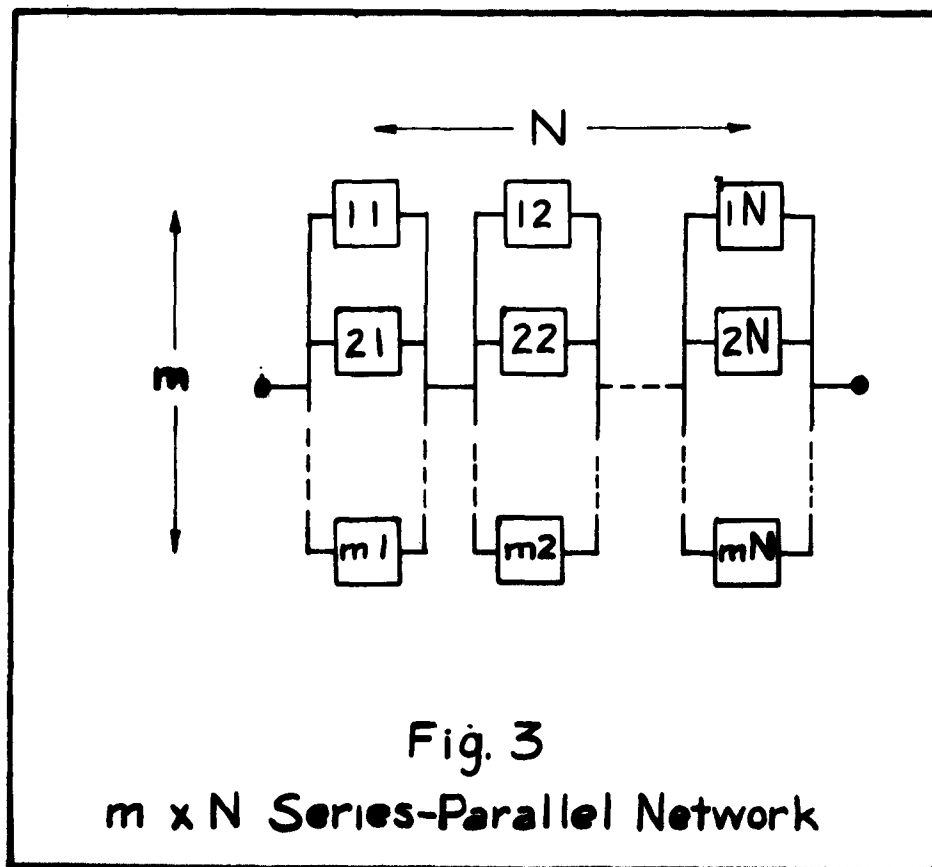
OPTIMUM NUMBER OF ELEMENTS FOR PARALLEL OR  
SERIES UNITS WHOSE ELEMENTS CAN SHORT AND OPEN

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Equating the right side to zero and solving for M,

$$M = \frac{\ln \left\{ \frac{\ln [1 - (1 - q_0)^n]}{\ln [1 - (q_0)^n]} \right\}}{\ln \left\{ \frac{[1 - (q_0)^n]}{[1 - (1 - q_0)^n]} \right\}} \quad (17)$$

Attention is now directed to the  $m \times N$  series-parallel network shown in Fig. 3. This network, like



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the previously discussed parallel-series network, is composed of elements which can fail in either the open or short-circuit failure mode. It is again assumed that the elements fail independently with the conditions of network failure being (a) at least one element in every unit fails short or (b) every element in at least one unit fails open. As before, (a) and (b) above will be referred to as events A and B respectively.

Since the development of the expression for the optimum number of units, N, for the series-parallel network is quite similar to that for the parallel-series network it will be described symbolically for the most part.

$$P(\text{no element fails short in unit } j) = \prod_{i=1}^m (1 - q_{si,j}) \quad (18)$$

$$P(\text{at least one element shorts in unit } j) = 1 - \prod_{i=1}^m (1 - q_{si,j}) \quad (19)$$

$$\text{so } P(A) = \prod_{j=1}^N \left[ 1 - \prod_{i=1}^m (1 - q_{si,j}) \right] \quad (20)$$

$$P(\text{all in unit } j \text{ fail open}) = \prod_{i=1}^m q_{oi,j} \quad (21)$$

$$P(\text{at least one in unit } j \text{ does not fail open}) = 1 - \prod_{i=1}^m q_{oi,j} \quad (22)$$

$$\begin{aligned} P(\text{at least one in all } N \text{ units does not fail open}) \\ = \prod_{j=1}^N \left[ 1 - \prod_{i=1}^m q_{oi,j} \right] \end{aligned} \quad (23)$$

and

$$P(B) = 1 - \prod_{j=1}^N (1 - \prod_{i=1}^m q_{oij}) \quad (24)$$

Therefore,

$$Q_N = P(A) + P(B) \quad (25)$$

$$Q_N = 1 - \prod_{j=1}^N (1 - \prod_{i=1}^m q_{oij}) + \prod_{j=1}^N [1 - \prod_{i=1}^m (1 - q_{oij})] \quad (26)$$

and since  $R_N = 1 - Q_N$ , then

$$R_N = \prod_{j=1}^N [1 - \prod_{i=1}^m q_{oij}] - \prod_{j=1}^N [1 - \prod_{i=1}^m (1 - q_{oij})] \quad (27)$$

For an equal element network, Eq (27) becomes

$$R_N = [1 - (q_o)^m]^N - [1 - (1 - q_o)^m]^N \quad (28)$$

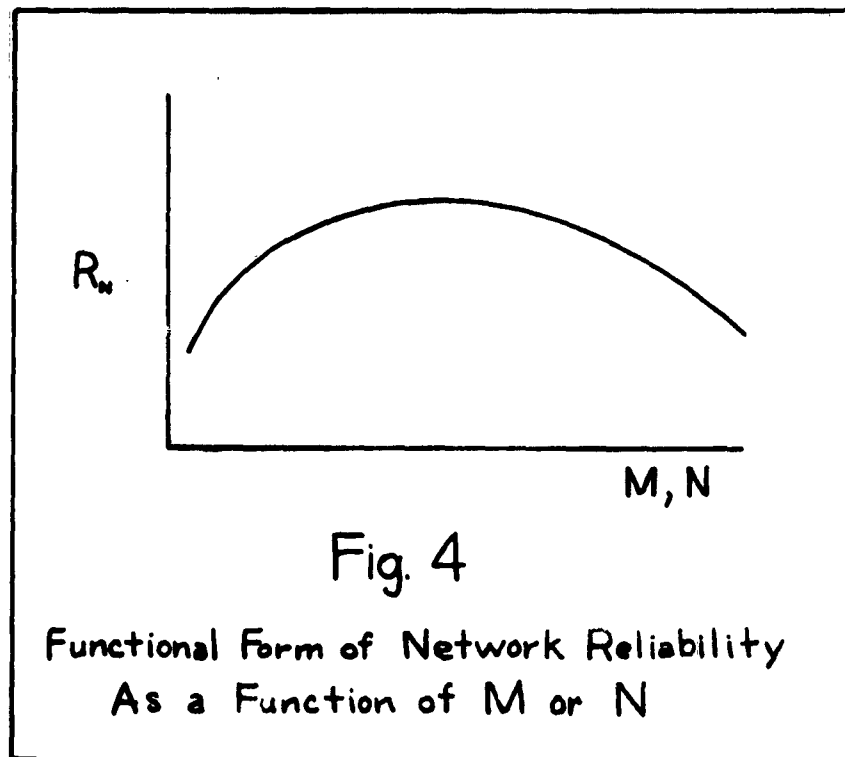
Again, to determine the number of units which will result in maximum network reliability, the derivative of  $R$  is taken with respect to  $N$ , equated to zero, and solved for  $N$ , resulting in the following equation.

$$N = \frac{\ln \left\{ \frac{\ln [1 - (1 - q_s)^m]}{\ln [1 - (q_s)^m]} \right\}}{\ln \left\{ \frac{[1 - (q_s)^m]}{[1 - (1 - q_s)^m]} \right\}} \quad (29)$$

The determining factor in the choice between the parallel-series or series-parallel network is the relative magnitude of  $q_s$  and  $q_p$ . If  $q_s > q_p$  then the elements should be connected in series. Conversely, if  $q_p > q_s$ , the parallel connection should be used. Therefore, when  $q_s > q_p$  Eq (17) for parallel-series networks must be used and when  $q_p > q_s$  Eq (29) for series-parallel networks is the correct equation.

In general, for given  $q_s$  and  $q_p$  network reliability as a function of M or N would have the form shown in Fig. 4. Clearly, since the curve is continuous, rational values of M or N are possible while only integral values of M are meaningful.

When the solution of Eq (17) or (29) results in other than integral values for M or N, further computation is necessary. The integers on either side of the non-integral value of M or N must be substituted in Eq (14) to determine the maximum obtainable network reliability.



The following steps outline the procedure to be used in applying the equations developed in this chapter to determine the optimum dimensions of parallel-series and series-parallel networks.

1. From given values of  $q_s$  and  $q_p$ , determine whether elements should be connected in parallel or series.
  - a. Parallel if  $q_s > q_p$ .
  - b. Series if  $q_p > q_s$ .
2. Find optimum number of elements from Fig. 2.

3. Substituting these values of  $q_s$ ,  $q_o$ , and  $m$  or  $n$  in Eq (17) or (29), solve for  $M$  or  $N$ .
4. Substitute the closest integers above and below  $M$  or  $N$  into Eq (14) or (20) and solve for  $R_N$ .
5. Maximum value of  $R_N$  obtained in step 4 above is the maximum obtainable reliability using the given elements.

To illustrate the above procedure, suppose a network is to be constructed of elements for which  $q_s = .10$  and  $q_o = .0004$ . With these probabilities of failure in the short and open failure modes respectively it is obvious that the elements should be connected in series to maximize reliability, and from Fig. 2 it is seen that the optimum number of these elements in series is four.

Substituting these values in Eq (17) and solving for  $M$ ,

$$M = \frac{\ln \left\{ \frac{\ln [1 - (1 - .0004)^4]}{\ln [1 - (.1)^4]} \right\}}{\ln \left\{ \frac{[1 - (.1)^4]}{[1 - (1 - .0004)^4]} \right\}} \quad (30)$$

and

$$M = 1.72 \quad (31)$$

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Since 1.72 series connected paths of four elements each has no meaning, it must be decided whether to use one or two paths. Solving Eq (14) for  $M = 1$ ,

$$R_n = [1 - (0.1)^4] - [1 - (1 - 0.0004)^4] \quad (32)$$

$$R_n = 0.9983 \quad (33)$$

and for  $M = 2$ ,

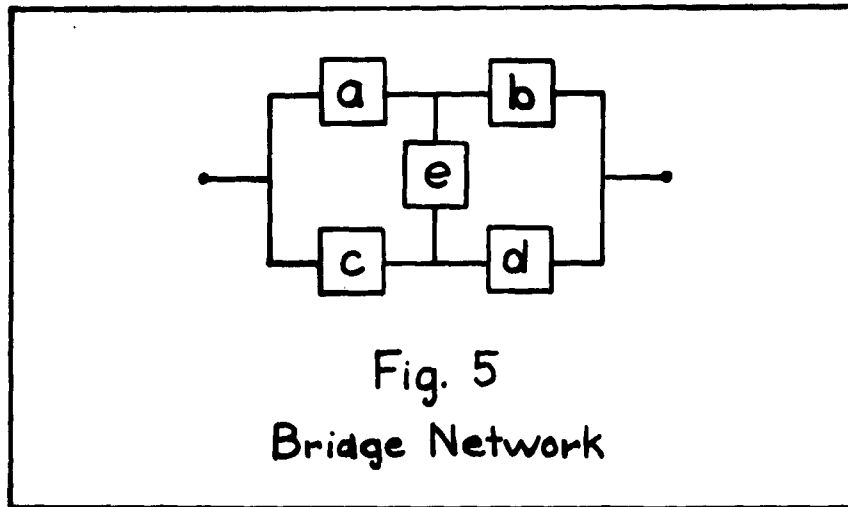
$$R_n = [1 - (0.1)^4]^2 - [1 - (1 - 0.0004)^4]^2 \quad (34)$$

$$R_n = 0.99979744 \quad (35)$$

Thus it is seen that two parallel paths of four elements each will provide the maximum reliability obtainable using the given elements in parallel-series or series-parallel networks.

III. The Bridge Network

The next network configuration to be considered is the bridge network shown in Fig. 5. Shannon and Moore have developed the reliability function for this



network under the assumption of equal elements which fail independently of each other ( Ref 1:195 ). This function is shown in Eq (36).

$$R_n = 2r^2 + 2r^3 - 5r^4 + 2r^5 \quad (36)$$

There are many occasions, however, when the networks under consideration might be composed of unlike elements and for these occasions it was thought useful to have more general expressions for network reliability in terms

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of element reliability. The basic assumptions for the following development are that the branch elements fail independently and that there is a single failure mode for each element.

An adaptation of the Disjunctive Expansion Theorem of Boolean Algebra ( Ref 2:xix ) to circuit analysis is used to develop the reliability expression for this and subsequent networks. This theorem states that every Boolean function,  $f(X_1, X_2, \dots, X_n)$ , may be written in the form:

$$f(X_1, X_2, \dots, X_n) \equiv f(1, X_2, \dots, X_n)X_1 \vee f(0, X_2, \dots, X_n)\bar{X}_1 \quad (37)$$

The assumption of a single mode of failure permits the application of Boolean Algebra to this network analysis.

In adapting this theorem to reliability calculations of complex networks it may be stated as follows: The reliability of a network is equal to the reliability factor of any one single element of the network times the reliability of the network with the terminals of the element shorted, plus the unreliability factor of the same element times the reliability of the network with the terminals of the element open. Or, expressed symbolically,

$$R_n = r_n [R_n]_{r_n=1} + q_n R_n \quad r_n=0 \quad (38)$$

where

$$R_n = \text{Network reliability} \quad (39)$$

$$r_n = \text{Reliability factor of element } K \quad (40)$$

$$Q_K = \text{Unreliability factor of element K.} \quad (41)$$

$$[R_N]_{r_K=1} = \text{Reliability of network if reliability of element K is one.} \quad (42)$$

$$[R_N]_{r_K=0} = \text{Reliability of network if reliability of element K is zero.} \quad (43)$$

When practical, it is helpful to sketch the network in each step of the development of the reliability expression. In order to simplify the notation in the development of the network reliability expression the following convention will be used,

$$Q = \text{reliability factor of element a.} \quad (44)$$

$$\bar{Q} = \text{unreliability factor of element a.} \quad (45)$$

Applying the factoring theorem to the bridge network of Fig. 5,

$$R_N = e \left[ - \left[ \begin{array}{c|c} a & b \\ \hline c & d \end{array} \right] \right] + \bar{e} \left[ - \left[ \begin{array}{c|c} a & b \\ \hline c & d \end{array} \right] \right] \quad (46)$$

Since the reliability expressions for the simplified circuits remaining in the brackets are known, these can now be substituted for the circuit diagrams, so that

$$R_n = e [(1 - \bar{a}\bar{c})(1 - \bar{b}\bar{d})] + \bar{e} [ab + cd - abcd] \quad (47)$$

and after expanding and grouping

$$R_n = e [1 - (\bar{a}\bar{c} + \bar{b}\bar{d} - \bar{a}\bar{b}\bar{c}\bar{d})] + \bar{e} [ab + cd - abcd] \quad (48)$$

The expression in the first bracket is the probability that at least one element is good in each of the pairs  $ac$  and  $bd$ , and that in the second bracket is the probability that at least one of the pairs  $ab$  and  $cd$  has both elements good. The two terms, then, are conditional probabilities, the first being the probability that the network will be successful given that the bridge element is good and the second being the probability that the network will be successful given that the bridge element is bad.

Since the two events for which the above are the probabilities of occurrence are exhaustive and mutually exclusive, Eq (48) is thus the reliability expression for the bridge circuit of Fig. 5.

By replacing the element unreliability factors with their i's complements and expanding the resulting expression the network reliability expression containing only element reliability factors can be obtained. This result is shown in Eq (49).

$$\begin{aligned}
 R_n = & ab + cd + aed + ceb - abcd \\
 & - abce - abde - acde \\
 & - bcde + abcde + abcde
 \end{aligned} \tag{49}$$

While it is easily seen that the first four terms in Eq (49) include all possible paths through the bridge circuit, it is not so apparent that the last two terms eliminate any duplication which may be contained in the first five. This fact can be verified, however, by use of the Venn diagram shown in Fig. 6. The circles, a,b,c,d, and e, correspond to the events that the respective elements are good and the small sub-areas which are involved in the bridge circuit have been numbered from one to thirteen simply for convenient reference. The terms in Eq (49) therefore, represent various combinations of the sub-areas of the Venn diagram as follows.

$$ab = 1 + 2 + 6 + 7 + 8 + 11 + 12 \tag{50}$$

$$cd = 3 + 4 + 8 + 9 + 10 + 11 + 13 \tag{51}$$

$$abcd = 8 + 11 \tag{52}$$

$$aed = 5 + 6 + 10 + 11 \tag{53}$$

$$ceb = 7 + 9 + 11 \tag{54}$$

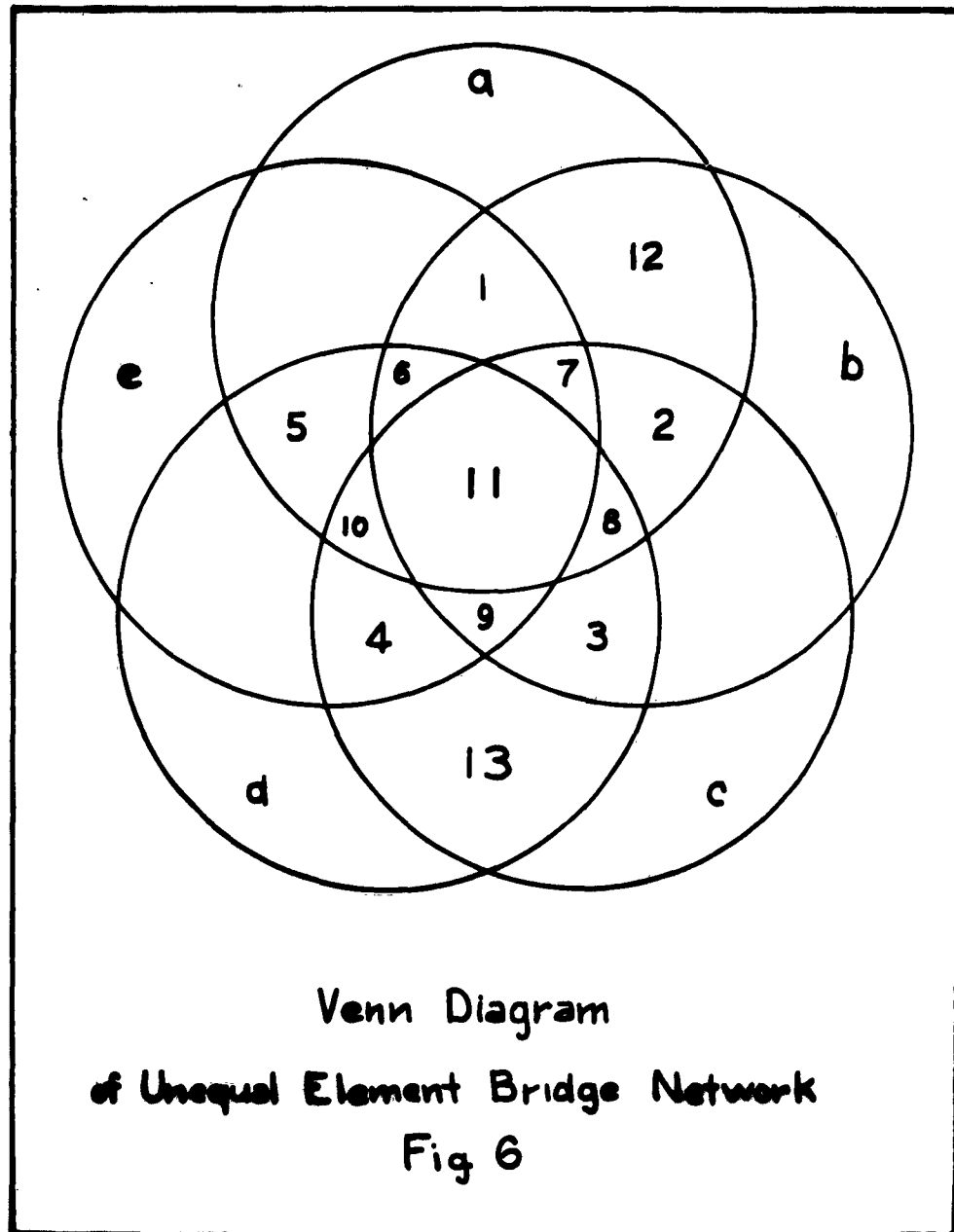
$$abce = 7 + 11 \tag{55}$$

$$abde = 6 + 11 \tag{56}$$

$$acde = 10 + 11 \tag{57}$$

$$bcde = 9 + 11 \tag{58}$$

$$abcde = 11 \tag{59}$$



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Writing Eq (49) in terms of these sub-areas

$$\begin{aligned}
 R_u = & (1+2+6+7+8+11+12) \\
 & + (3+4+8+9+10+11+13) \\
 & + (5+6+10+11) + (7+9+11) \\
 & - (8+11) - (7+11) - (6+11) \\
 & - (10+11) - (9+11) + 11 + 11
 \end{aligned} \tag{60}$$

$$\begin{aligned}
 R_u = & 1+2+6+7+8+11+12+3+4 \\
 & + \cancel{8}+9+10+\cancel{11}+13 - \cancel{8} - \cancel{11} + 5 \\
 & + \cancel{8} + \cancel{10} + \cancel{11} + \cancel{7} + \cancel{9} + \cancel{11} - \cancel{7} - \cancel{11} - \cancel{8} \\
 & - \cancel{11} - \cancel{10} - \cancel{11} - \cancel{8} - \cancel{11} + \cancel{11} + \cancel{11}
 \end{aligned} \tag{61}$$

$$\begin{aligned}
 R_u = & 1+2+3+4+5+6+7+8+9 \\
 & + 10+11+12+13
 \end{aligned} \tag{62}$$

and thus it is seen that there is no duplication present in Eq (49).

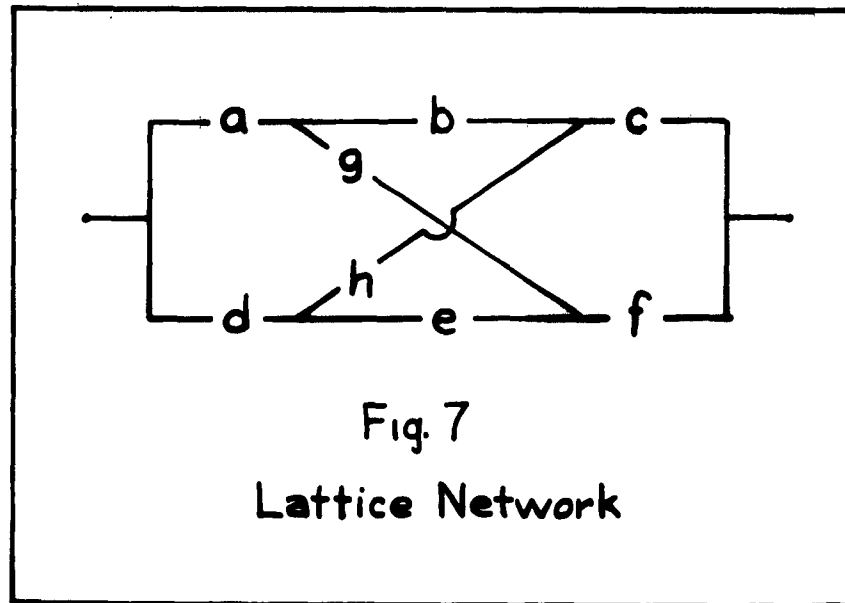
Eq (49) is rather unwieldy for the computation of reliability, either by hand or by computer programming. Suitable grouping and factoring of the terms will lead to Eq (63) which is the sum of two terms minus their product. Eq (63) would be relatively simple to program for computer solution.

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$$R_n = a[b + ed - bed] + c[d + eb - deb] \\ - ac[b + ed - bed][d + eb - deb] \quad (63)$$

#### IV. The Lattice Network

The eight element lattice network, frequently encountered in computer logic circuits, is the next one to be considered. A drawing of this type of network is shown in Fig. 7.



The basic assumptions in the development of the reliability expression for this network are that the elements fail independently and that each has only one failure mode. The expression will be developed first for a lattice composed of diode type elements; that is, flow in the network is from left to right only. Again, the factoring theorem may be used to get

$$R_n = a \left[ \begin{array}{c} \text{---} b \text{---} c \\ \diagup \quad \diagdown \\ d \quad e \quad f \end{array} \right] + \bar{a} \left[ -d \begin{array}{c} \text{---} h \text{---} c \\ \text{---} e \text{---} f \end{array} \right] \quad (64)$$

While elements  $b$  and  $g$  might be good with element  $a$  bad, as in the second term, they are useless under the assumption of left to right flow only. They are omitted, therefore, from the second term leaving it in a form requiring no further simplification, of the first term.

Continuing the simplification of the first term,

$$R_n = ad \left[ \begin{array}{c} \text{---} h \text{---} c \\ \text{---} b \text{---} \\ \text{---} g \text{---} f \end{array} \right] + a\bar{d} \left[ \begin{array}{c} \text{---} b \text{---} c \\ \text{---} g \text{---} f \end{array} \right] + \bar{a}d \left[ \begin{array}{c} \text{---} h \text{---} c \\ \text{---} e \text{---} f \end{array} \right] \quad (65)$$

All three brackets now contain circuits composed of simple series and/or parallel arrangements of the elements and may be replaced with suitable algebraic expressions. Thus

$$R_n = ad \left[ c(1 - \bar{b}\bar{h}) + f(1 - \bar{e}\bar{g}) - cf(1 - \bar{b}\bar{h})(1 - \bar{e}\bar{g}) \right] + a\bar{d} [bc + fg - bcf g] + \bar{a}d [ch + ef - cefh] \quad (66)$$

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which is the general expression for the reliability of a lattice network in terms of both reliability and unreliability factors.

Replacing the unreliability factors with their 1's complements, expanding, and collecting terms gives

$$\begin{aligned} R_n = & abc + afg - abcfg + cdh + def \\ & - cdefh - abcdh - adefg \\ & - adcfgh + acdefgh - abcdef \\ & + abcdefg + abcdefh \\ & + abcd fgh - abcdefgh \end{aligned} \quad (67)$$

By grouping and factoring the first three and the second three terms, the two terms  $a[bc+fg-bcfg]$  and  $d[ch+ef- ch ef]$  are obtained. Noticing that elements  $a$  and  $d$  are "in parallel" in Fig. 6 and that the first of the two terms just obtained is the reliability of the lattice if element  $a$  is good and that the second is the reliability of the lattice if element  $d$  is good, raises the question whether the last nine terms might not be the result of the negative product of the first two terms. If this were true we would have the simple parallel relationship involving the sum of two expressions minus their product. To establish whether or not this is the case we take the negative product of the first two terms, expand, and check the results against the

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last nine terms of Eq (67).

$$\begin{aligned}
 & -ad[bc + fg - bcfg][ch + ef - cehf] \\
 & = -ad[bcch + bcef - bccehf \\
 & \quad + cfgh + effg - ceffgh \\
 & \quad - bccfgh - bceffg \\
 & \quad + bcceffgh] \tag{68}
 \end{aligned}$$

Since the letters representing the network elements stand for the events that the elements are good, we can apply the idempotent law of Boolean Algebra ( Ref 2:xiii) to the above expression to eliminate the double factors which appear in some of the terms. Continuing the expansion,

$$\begin{aligned}
 & -ad[bc + fg - bcfg][ch + ef - cefh] \\
 & = -ad[bch + bcef - bcefh \\
 & \quad + cfgh + efgh - cefgh \\
 & \quad - bcfgh - bcefg \\
 & \quad + bceffgh] \tag{69}
 \end{aligned}$$

$$\begin{aligned}
 & = -abcdh - abcdef + abcdefh \\
 & \quad - acdfgh - adefg + acdefgh \\
 & \quad + abcdffgh + abcdefg \\
 & \quad - abcdefgh \tag{70}
 \end{aligned}$$

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These are identically the last nine terms of Eq (67) which may now be written

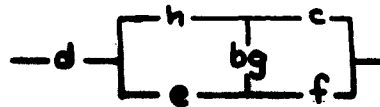
$$R_n = a [bc + fg - bcfq] + d [ch + ef - cefh] - ad [bc + fg - bcfq] [ch + ef - cefh] \quad (71)$$

Eq (71) will be much easier to use in calculating reliabilities than Eq (67), either by desk calculator or computer program.

Sometimes the elements or branches in a network permit flow in either direction, so the restriction of one-way flow imposed upon the network elements in the development of Eq (67) is now lifted for the following development. Again applying the factoring theorem to the network of Fig. 7,

$$R_n = a \left[ \text{circuit diagram} \right] + \bar{a} \left[ \text{circuit diagram} \right] \quad (72)$$

Because of the assumption of two-way flow, elements  $b$  and  $q$  in the second bracket cannot now be disregarded. The circuit can, however, be drawn as element  $d$  in series with a bridge circuit.



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With the reliability expression for the bridge circuit now available no further simplification of this circuit is necessary.

Continuing,

$$R_n = ad \left[ \begin{array}{c} \text{---} \overline{h} \text{---} c \\ | \\ \text{---} \overline{b} \text{---} \\ | \\ \text{---} \overline{g} \text{---} f \\ | \\ \text{---} \overline{e} \text{---} \end{array} \right] + a\bar{d} \left[ \begin{array}{c} \text{---} b \text{---} c \\ | \\ \text{---} \overline{h} e \text{---} \\ | \\ \text{---} g \text{---} f \\ | \\ \text{---} \end{array} \right] + \bar{a}d \left[ \begin{array}{c} \text{---} h \text{---} c \\ | \\ \text{---} \overline{b} g \text{---} \\ | \\ \text{---} e \text{---} f \\ | \\ \text{---} \end{array} \right] \quad (73)$$

Eq (72) has now been reduced to the point where algebraic expressions may be substituted for all the circuits in the brackets. Eq (73) is, in fact, the same as Eq (65) with the two parallel-series circuits of Eq (65) bridged by the elements missing in those two terms.

In using the bridge network reliability expression developed in Chapter III it is helpful to construct a table of corresponding terms.

$\left[ \begin{array}{c} \text{---} a \text{---} b \\   \\ \text{---} e \text{---} \\   \\ \text{---} c \text{---} d \\   \\ \text{---} \end{array} \right]$	$\left[ \begin{array}{c} \text{---} b \text{---} c \\   \\ \text{---} \overline{h} e \text{---} \\   \\ \text{---} g \text{---} f \\   \\ \text{---} \end{array} \right]$	$\left[ \begin{array}{c} \text{---} h \text{---} c \\   \\ \text{---} \overline{b} g \text{---} \\   \\ \text{---} e \text{---} f \\   \\ \text{---} \end{array} \right]$
a	b	h
b	c	c
c	g	e
d	f	f
e	he	bg

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Substituting the corresponding terms into Eq (73)  
the expression

$$\begin{aligned} & \left[ bc + fg + bef h + ceg h - bcf g - bcegh \right. \\ & \quad \left. - bcefh - befgh - cefgh \right. \\ & \quad \left. + bcefg h + bce fgh \right] \end{aligned}$$

is obtained for the first bridge and

$$\begin{aligned} & \left[ hc + ef + bfg h + bce g - cef h \right. \\ & \quad \left. - bcegh - bcfgh - befgh \right. \\ & \quad \left. - bcefg + bce fgh + bce fgh \right] \end{aligned}$$

is obtained for the second bridge. Eq (73) may now  
be written

$$\begin{aligned} R_u = & ad \left[ c(1 - \bar{h}\bar{b}) + f(1 - \bar{e}\bar{g}) \right. \\ & \quad \left. - cf(1 - \bar{h}\bar{b})(1 - \bar{e}\bar{g}) \right] \\ & + a\bar{d} \left[ bc + fg + bef h + ceg h - bcf g \right. \\ & \quad \left. - bcegh - bcefh - befgh \right. \\ & \quad \left. - cefgh + bce fgh + bce fgh \right] \\ & + \bar{a}d \left[ hc + ef + bfg h + bce g - cef h \right. \\ & \quad \left. - bcegh - bcfgh - befgh \right. \\ & \quad \left. - bcefg + bce fgh + bce fgh \right] \end{aligned} \quad (74)$$

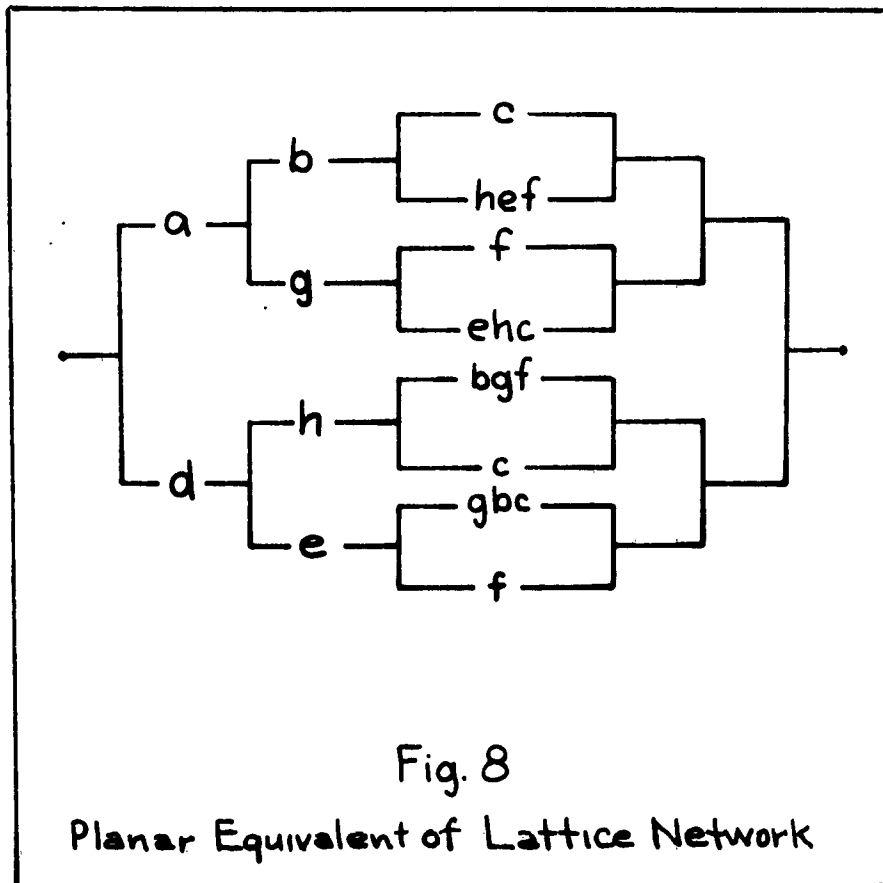
After replacing the unreliability factors with their 1's complements, expanding, and collecting terms, the following expression for the reliability of the lattice network in terms of element reliability factors is obtained.

$$\begin{aligned}
 R_n = & abc + afg + cdh + def - abcdh \\
 & - abcfg - adefg - cdefh \\
 & + abefh + acegh + bcdeg + bdfgh \\
 & - abcdef - abcdeg - abcefh \\
 & - abcegh - abdefh - abdfgh \\
 & - abefgh - acdegh - acdfgh \\
 & - acefgh - bcdefg - bcdegh \\
 & - bcd fgh - bde fgh + 2 abcdefg \\
 & + 2 abcdefh + 2 abcdegh + 2 abcd fgh \\
 & + 2 abc e fgh + 2 abdefgh + 2 acdefgh \\
 & + 2 bcdefgh - 5 abcdefgh
 \end{aligned} \tag{75}$$

While Eq (75) is in the simplest possible expanded form it is still somewhat unwieldy. In order to facilitate its use in desk calculator computations, it should be noted that certain combinations of factors appear throughout the expression, and that by tabulating the values of these recurring combinations initially, these combination factor values can be used repeatedly minimizing the total required multiplications. The combinations  $abc$  and  $fgh$ , for example, appear in thirteen of the terms, and  $de$  appears in eighteen.

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The form of Eq (75) is especially ill-suited for a minimum statement computer program. To provide a form of the lattice network reliability expression which will permit a minimum statement program of orderly sequential arithmetic operations to be written, the nonplanar circuit of Fig. 7 is first transformed into a planar one by duplicating some of the elements ( Ref 4:60,68 ). Fig. 8 is a drawing of the resulting planar circuit.



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The twenty-two element planar network of Fig. 8 is the logical equivalent of the eight element non-planar network of Fig. 7, and in this form the elements are seen to have pyramided parallel relationships. The reliability expression for the network of Fig. 8 is given by

$$\begin{aligned} R_n = & a \left[ b(c + efh - cefh) \right. \\ & + g(f + ceh - cefh) \\ & \left. - bg(c + efh - cefh)(f + ceh - cefh) \right] \\ & + d \left[ e(f + bcg - bcfq) \right. \\ & + h(c + bfg - bcfq) \\ & \left. - eh(f + bcg - bcfq)(c + bfg - bcfq) \right] \\ & - ad \left[ b(c + efh - cefh) \right. \\ & + g(f + ceh - cefh) \\ & \left. - bg(c + efh - cefh)(f + ceh - cefh) \right] \\ & \times \left[ e(f + bcg - bcfq) \right. \\ & + h(c + bfg - bcfq) \\ & \left. - eh(f + bcg - bcfq)(c + bfg - bcfq) \right] \quad (76) \end{aligned}$$

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Proof that the networks Fig. 7 and Fig. 8 and the network reliability expressions of Eq (75) and Eq (76) are equivalent is given by the fact that if Eq (76) is expanded, Eq (75) is obtained identically.

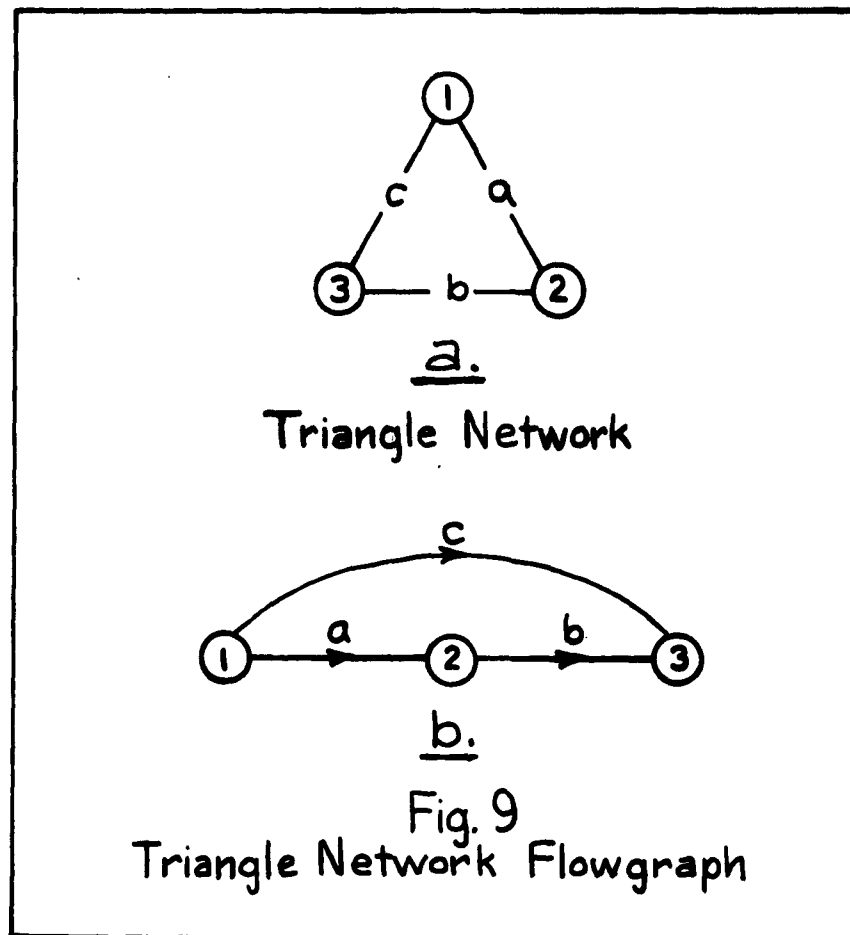
At first glance Eq (76) may not appear to be less formidable than Eq (75). However, the last large term is simply the negative product of the first two smaller ones, and within the brackets of the first two terms the larger third term is again the negative product of the first two smaller ones, each of which contains the sum of two terms and their negative product. The ability of the digital computer to perform an arithmetic operation and store the result for later use in other arithmetic operations will permit Eq (76) to be programmed with relatively few statements.

## V. Polygonal Networks

In this chapter the reliability of a polygonal class of networks will be investigated. These networks will consist of two or more vertices or stations each connected with all the others by direct paths along which information may flow, the stations being designated by numbers and the paths by letters. The reliability of any path will be the probability of successful transmission of information between the two stations connected by the path while the reliability of the network will be the probability of successful transmission of information between any two stations in the network by any sequence of paths. It is assumed that the paths fail independently of each other with a single failure mode.

The first case is the trivial one of two stations with one connecting path and is mentioned only because it will be compared with the other networks. The network reliability is, of course, the path reliability.

The three station network has three paths as shown in Fig. 9.a. Since the use of flow graphs will be very helpful in the large networks, their use will be introduced at this time. The flow graph depicting the flow of information from station one to station three is shown in Fig. 9.b.



Network reliability for transmitting information from station one to station three is thus seen to be

$$R_{N_3} = c + ab - abc \quad (77)$$

the subscript three referring to the number of stations in the network. For convenience at this time

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it will be assumed that the paths have equal reliabilities,  $r$ , so that

$$R_{n_1} = r + r^2 - r^3 \quad (78)$$

for the three station network.

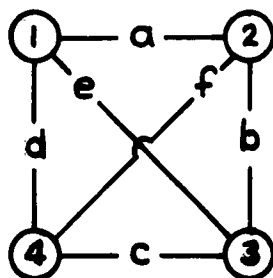
The network drawing and a flow graph of the four station network between stations one and two are shown in Fig. 10.a and 10.b, respectively. From the flow graph in Fig. 10.b, the network circuit may be drawn as shown in Fig. 10.c, a direct path in parallel with a bridge circuit. Making use of Eq (36), Shannon's and Moore's equal element bridge reliability function ( Ref 1:195 ), the four station network reliability function can be written as

$$\begin{aligned} R_{n_4} = & r + 2r^2 + 2r^3 - 5r^4 + 2r^5 \\ & - r(2r^2 + 2r^3 - 5r^4 + 2r^5) \end{aligned} \quad (79)$$

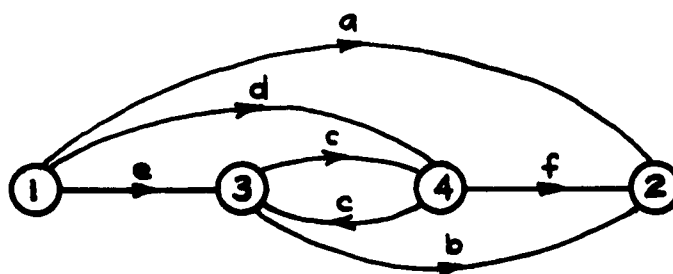
which reduces to

$$R_{n_4} = r + 2r^2 - 7r^4 + 7r^5 - 2r^6 \quad (80)$$

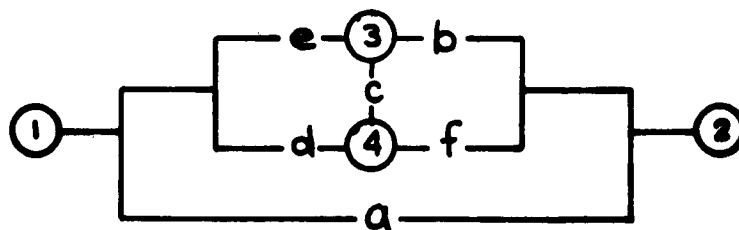
The pentagonal network, flow graph and circuit diagram are shown in Fig. 11.a, 11.b, and 11.c, respectively. The upper portion of the circuit diagram is seen to be nonplanar and so the disjunctive expansion theorem must be used to simplify this part of the circuit.



a. Square Network

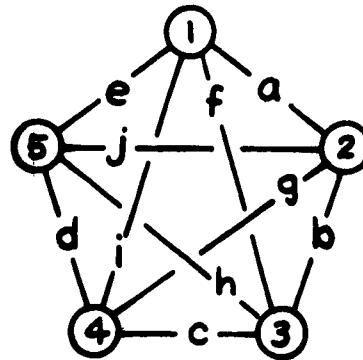


b. Square Network Flowgraph

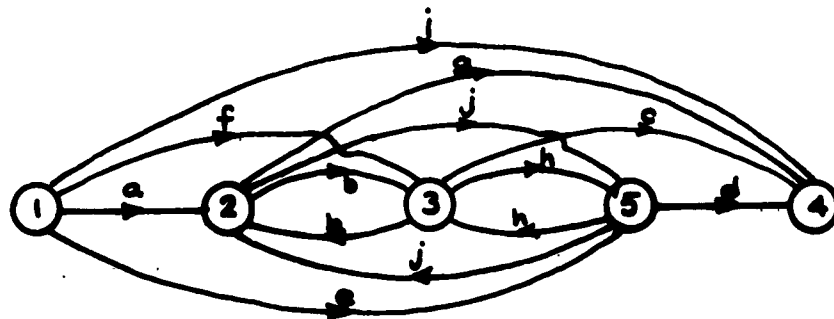


c. Square Network Circuit Diagram

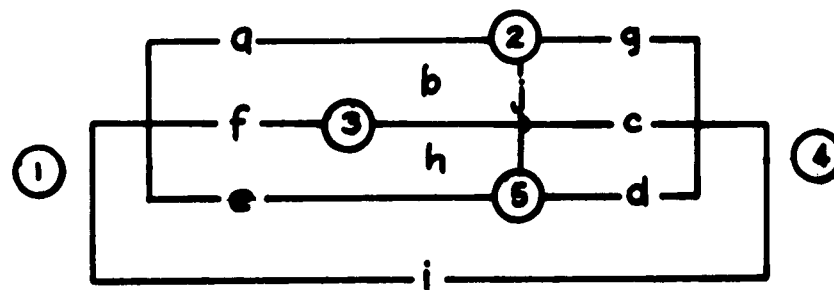
Fig. 10



a. Pentagonal Network



b. Pentagonal Network Flowgraph



c. Pentagonal Network Circuit Diagram  
Fig 11

$$R_{N_5} = j \left[ \begin{array}{c} \begin{array}{ccc} a & \begin{array}{c} \diagup b \diagdown \\ \hline \end{array} & g \\ f & \begin{array}{c} \diagdown h \diagup \\ \hline \end{array} & c \\ e & & d \end{array} \end{array} \right] + \bar{j} \left[ \begin{array}{c} \begin{array}{ccc} a & b & g \\ f & b & c \\ e & h & d \end{array} \end{array} \right] \quad (81)$$

$$R_{N_5} = j b \left[ \begin{array}{c} \begin{array}{ccc} a & \begin{array}{c} \diagup \diagdown \\ \hline \end{array} & g \\ f & \begin{array}{c} \diagdown \diagup \\ \hline \end{array} & c \\ e & & d \end{array} \end{array} \right] + j \bar{b} \left[ \begin{array}{c} \begin{array}{ccc} a & & g \\ f & & c \\ e & & d \end{array} \end{array} \right] + \bar{j} b \left[ \begin{array}{c} \begin{array}{ccc} a & & g \\ f & & c \\ e & h & d \end{array} \end{array} \right] + \bar{j} \bar{b} \left[ \begin{array}{c} \begin{array}{ccc} a & & g \\ f & & c \\ e & h & d \end{array} \end{array} \right] \quad (82)$$

$$R_{N_5} = j b \left[ \begin{array}{c} \begin{array}{cc} \begin{array}{c} a \\ f \\ e \end{array} & \begin{array}{c} g \\ c \\ d \end{array} \end{array} \right] + j \bar{b} h \left[ \begin{array}{c} \begin{array}{cc} \begin{array}{c} a \\ f \\ e \end{array} & \begin{array}{c} g \\ c \\ d \end{array} \end{array} \right] + j \bar{b} h \left[ \begin{array}{c} \begin{array}{cc} \begin{array}{c} a \\ e \\ f \end{array} & \begin{array}{c} g \\ d \\ c \end{array} \end{array} \right] + \bar{j} b h \left[ \begin{array}{c} \begin{array}{cc} \begin{array}{c} a \\ f \\ e \end{array} & \begin{array}{c} g \\ c \\ d \end{array} \end{array} \right] + \bar{j} b h \left[ \begin{array}{c} \begin{array}{cc} \begin{array}{c} a \\ f \\ e \end{array} & \begin{array}{c} g \\ c \\ d \end{array} \end{array} \right] + \bar{j} \bar{b} h \left[ \begin{array}{c} \begin{array}{cc} \begin{array}{c} a \\ f \\ e \end{array} & \begin{array}{c} g \\ c \\ d \end{array} \end{array} \right] + \bar{j} \bar{b} \left[ \begin{array}{c} \begin{array}{ccc} a & & g \\ f & & c \\ e & h & d \end{array} \end{array} \right] \quad (83)$$

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The circuit diagrams in Eq (83) can now be replaced with algebraic expressions to obtain, for equal paths,  $q$  being the path unreliability factor,

$$\begin{aligned}
 R_{N_1} = & r^2[(1-q^3)(1-q^3)] + r^2q[(1-q^3)(1-q^3)] \\
 & + r^2q^2[r^2 + (1-q^3)(1-q^3) - r^2(1-q^3)(1-q^3)] \\
 & + r^2q[(1-q^3)(1-q^3)] \\
 & + r^2q^2[r^2 + (1-q^3)(1-q^3) - r^2(1-q^3)(1-q^3)] \\
 & + q^2[r^2 + 2r^2 + 2r^3 - 5r^4 + 2r^5 \\
 & \quad - r^2(2r^2 + 2r^3 - 5r^4 + 2r^5)] \quad (84)
 \end{aligned}$$

Regrouping and factoring

$$\begin{aligned}
 R_{N_1} = & [r^2 + 2r^2q][(1-q^3)(1-q^3)] \\
 & + 2r^2q^2[r^2 + (1-q^3)(1-q^3) - r^2(1-q^3)(1-q^3)] \\
 & + q^2[r^2 + 2r^2 + 2r^3 - 5r^4 + 2r^5 \\
 & \quad - r^2(2r^2 + 2r^3 - 5r^4 + 2r^5)] \quad (85)
 \end{aligned}$$

which gives, after expanding and collecting terms,

$$\begin{aligned}
 R_{N_1} = & 3r^2 + 6r^3 - 9r^4 - 36r^5 + 91r^6 \\
 & - 84r^7 + 36r^8 - 6r^9 \quad (86)
 \end{aligned}$$

The reliability of the pentagonal network is then given by

$$\begin{aligned}
 R_{N_1} = & r + 3r^2 + 3r^3 - 15r^4 - 27r^5 + 127r^6 \\
 & - 175r^7 + 120r^8 - 42r^9 + 6r^{10} \quad (87)
 \end{aligned}$$

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The complexity of polygonal networks of more than five vertices is such that the time required to solve them by the forgoing methods makes an alternate method desirable. While it is beyond the scope of this study, the technique of computer simulation should prove fruitful in this regard.

To compare the reliabilities of the polygonal networks, an element reliability factor of  $r = .9$  was used and the results are shown in Fig. 12. As can be seen, network reliability increased as more stations are added and rapidly approaches unity.


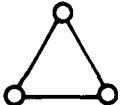
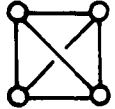
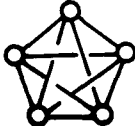
Network	Reliability Function	$R_N: r=0.9$
	$R_{N_2} = r$	0.90
	$R_{N_3} = r + r^2 - r^3$	0.9810
	$R_{N_4} = r + 2r^2 - 7r^4 + 7r^5 - 2r^6$	0.9978480
	$R_{N_5} = r + 3r^2 + 3r^3 - 15r^4 - 27r^5 + 127r^6 - 175r^7 + 120r^8 - 42r^9 + 6r^{10}$	0.9997948

Fig. 12

## VI. Summary and Conclusions

The purpose of this thesis was to develop reliability functions for the parallel-series, series-parallel, bridge, lattice, and polygonal networks and to formulate procedures for the use of these functions.

The functions for the parallel-series and series-parallel networks were developed assuming equal elements having the open and short-circuit failure modes. The functions are expressed in terms of network dimensions and permit designing a maximum reliability network for given elements. The procedure for their use in designing maximum reliability networks is given also.

The reliability functions for the bridge and lattice networks were developed for networks composed of unequal elements having a single failure mode. The reduced form of these functional expressions may be used for substitution into more complex network functions and the unreduced forms can be used in reliability calculations or computer programming.

The reliability functions of the first five of the polygonal class of networks were formulated for equal path reliabilities and network reliabilities calculated for  $r = .9$ . It was seen that highly reliable networks can be realized by interconnecting the stations as was done for the polygonal networks.

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It is recommended for further study that a simulation program be written for a general polygonal network of  $n$  vertices. It is also recommended that computer subroutines or short programs be written which will use the developed bridge and lattice reliability functions for computing network reliability.

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## Appendix A

Disjunctive Expansion Theorem of Boolean Algebra

The reliability of a redundant network is equal to the reliability factor of any one single element of the network times the reliability of the network with the terminal of the element shorted, plus the unreliability factor of the same element times the reliability of the net with the terminals of the element open.

$$R_N = r_K [R_N]_{r_K=1} + q_K [R_N]_{r_K=0} \quad (88)$$

Proof:

Let  $R_N$  = Reliability of a net expressible in canonical form.

Canonical form = the sum of all possible favorable probabilities from the canonical equation:  
 Canonical equation = all permutations of network elements.

Let  $H_a$  = the sum of those terms containing  $r_K$ .

$H_b$  = the sum of those terms containing  $q_K$ .

then

$$H_a = r_K H_s \quad (89)$$

$$H_b = q_K H_o \quad (90)$$

and

$$R_N = H_a + H_b = r_K H_s + q_K H_o \quad (91)$$

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Let  $r_k = 1$

$$[R_N]_{r_k=1} = H_s \quad (92)$$

and  $q_k = 1$ , or  $r_k = 0$

$$[R_N]_{r_k=0} = H_o \quad (93)$$

Substituting Eqs (92) and (93) into Eq (91)

$$R_N = r_k [R_N]_{r_k=1} + q_k [R_N]_{r_k=0} \quad (94)$$

which is Eq (88). Q.E.D.